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A NEW RELATIVE SKILL MEASURE FOR GAMES WITH CHANCE ELEMENTS

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Abstract

An interesting aspect of games is the relative extent to which a player can positively influence his results by making appropriate strategic choices. This question is closely related to the issue of how to distinguish between games of skill and games of chance. The distinction between these two types of games is definitely interesting from a juridical point of view.

Borm and Van der Genugten (2001) presented a method to measure the skill level of a game. In principle, their measure can serve as a juridical tool for the classification of games with respect to skill. In this paper we present a modification of the measure. The main difference is that this new definition does not automatically classify incomplete information games without chance moves as games of skill. We use a coin game and a simplified version of standard drawpoker as an illustration.

Keywords: games of skill; games of chance.
JEL code: C72.

1 Introduction

In various countries, including the Netherlands, legislation is such that the question whether a specific game should be considered as a game of chance or as a game of skill is predominant in the exploitation possibilities of private casinos. Borm and Van der Genugten (2001) developed an objective and operational criterion to quantify the level of skill in games with chance elements. This measure can serve as a juridical tool for the classification of games, since its definition is in accordance with the relevant paragraphs in the Dutch Gaming Act.

A recent article on skill in games is Larkey et al. (1997). Whereas we are interested in the skill involved in the game as a whole, this article focuses on the skill differences between

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players. Moreover, it provides an interesting discussion on the interpretation and relevance of the skill concept in analyzing games. A study of psychological aspects of chance and skill elements in games can be found e.g. in Keren and Wagenaar (1988).

By definition of the skill measure of Borm and Van der Genugten (2001), every game in which no *external* chance moves are present, is a pure game of skill. Examples of external chance moves are the dealing of cards and the spin of a roulette wheel. In more-person games a different kind of chance element, generated by the players who use mixed strategies, can play a role. We will refer to this type of chance moves as *internal* chance moves.

In this paper we redefine the skill measure, such that these internal chance moves are taken into account. This results in a modification for more-person games only. For one-person games the definition does not differ from the one given by Borm and Van der Genugten (2001). In section 2 we recall the definition of relative skill for one-person games. In section 3 we discuss the consequences of the original extension of the skill measure to more-person games and subsequently present the modified definition. In section 4 the theory is illustrated by two examples. We compute the skill level of two simple, but non-trivial games: a coin game and a poker game. Section 5 concludes.

2 One-person games

We consider a finite game for *one player*, possibly with chance moves. The number of chance moves must be finite and for each chance move the number of possible outcomes must be finite. The rules of the game determine a finite, non-empty set X of pure strategies of the player. For this game we also consider the *fictive* situation in which the player knows the outcomes of the chance moves before he has to make a decision. This knowledge extends the set of strategies to a finite set $\bar{X} \supseteq X$.

The quality of a strategy $\bar{x} \in \bar{X}$ is determined by its corresponding *expected gains*. The expectation of the player's gains with respect to the external chance moves is given by the function U that assigns expected gains $U(\bar{x})$ to each strategy $\bar{x} \in \bar{X}$. Let $\Delta(X)$ and $\Delta(\bar{X})$ denote the sets of all probability distributions on X and $\bar{X}$ respectively. Using expectations with respect to these internal chance elements, the extension $U(\bar{\sigma})$ for mixed strategies $\bar{\sigma} \in \Delta(\bar{X})$ is immediate.

The definition of (potential) *relative skill* is based on the expected gains of three *types* of players: the *beginner*, the *optimal player* and the *fictive player*. A beginner is associated with a given strategy $\sigma^0 \in \Delta(X)$ with corresponding expected gains

$$U^0 := U(\sigma^0).$$

The optimal player uses a strategy with maximal expected gains, i.e.

$$U^m := \max_{x \in X} U(x).$$

Clearly, the fact that the optimal player maximizes over his set of pure strategies, instead of its mixed extension, does not affect his maximum expected gains. We could also have written $U^m := \max_{\sigma \in \Delta(X)} U(\sigma)$. The fictive player has the extra information on the outcome of the chance moves and can do at least as good as the optimal player, but possibly better. He uses a strategy in $\bar{X}$ with maximal expected gains. We write

$$U^f := \max_{\bar{x} \in \bar{X}} U(\bar{x}).$$

Using the same arguments as for the optimal player, we could also have written $U^f := \max_{\bar{\sigma} \in \Delta(\bar{X})} U(\bar{\sigma})$. These definitions lead to a nice ordering of the expected gains of the three player types: $U^0 \leq U^m \leq U^f$. We call the difference between the expected gains of the optimal player and the beginner, $U^m - U^0$, the *learning effect* LE in the game, while we refer to the difference between the expected gains for fictive and optimal players, $U^f - U^m$, as the *random effect* RE in the game.

The definition of the relative skill level RS of the game is based on the ratio of the learning effect and the random effect:

$$RS = \frac{LE}{LE + RE} = \frac{U^m - U^0}{U^f - U^0}.$$

Clearly, both the learning effect and the random effect are nonnegative. As a consequence, $0 \leq RS \leq 1$. For RS to be equal to its lower bound, the learning effect must be zero. Therefore $RS = 0$ indicates a *pure game of chance*. On the other hand, we have $RS = 1$ if there is no random effect in the game. Therefore, this extreme case corresponds to a *pure game of skill*.

For the sake of completeness we define $RS = 1$ if $LE = RE = 0$. This boundary case is only of theoretical importance, because in practical games this will not occur. In a game with $LE = RE = 0$ the chance elements do not have a restrictive influence on the maximal expected gains a player can attain, but the game is so easy that even a beginner can figure out how to play optimally.

3 More-person games

The generalization of the definition of relative skill for one-person games to n-person games is not straightforward. To study skill in more-person games, we will use an objective analysis of strategic behaviour based on the assumption of rationality of players. We will follow the lines of game theory, the mathematical theory of conflict situations that was initiated by Von Neumann and Morgenstern (1944).

We consider a finite game with player set $N = \{1, \ldots, n\}$, again possibly with chance moves. The finite, non-empty set X_i contains the pure strategies of player i. The set of strategy profiles of the players is then $X := \prod_{i=1}^{n} X_i$. For each player i the fictive situation that he knows the outcomes of the chance moves leads to the extended set $\bar{X}_i \supseteq X_i$ of strategies. This leads to the extension $\bar{X} := \prod_{i=1}^{n} \bar{X}_i$. For player i, $\Delta(X_i)$ and $\Delta(\bar{X}_i)$ denote his sets of mixed strategies as a normal and as a fictive player respectively. The product sets $\prod_{i=1}^{n} \Delta(X_i)$ and $\prod_{i=1}^{n} \Delta(\bar{X}_i)$ contain all possible strategy profiles.

For all $i \in N$ the function U_i assigns to each strategy profile $\bar{x} \in \bar{X}$ the expected gains $U_i(\bar{x})$ of player i. The vector $U(\bar{x}) = (U_1(\bar{x}), \ldots, U_n(\bar{x}))$ specifies the gains of all players. The extension $U(\bar{\sigma})$ for mixed strategy profiles in the set $\prod_{i=1}^{n} \Delta(\bar{X}_i)$ is straightforward.

As before, we base our definition of *relative skill* on the expected gains of three types of players: the beginner, the optimal player and the fictive player. However, these types must now be defined for each *role* a player can take. After all, in most games player roles are not symmetric. The difficulty, compared to the one-player case, is that a player's gains may now depend on the strategy choices of his opponents. Borm and Van der Genugten (2001) solved this problem in the following way. For each player $i \in N$ the strategy choices of the other players are thought to be fixed. The uniform reference for the three player types in the role of player i is a minimax strategy of the coalition of all opponents of player i in the related two-person zero-sum game.

A drawback of this method is that the coalition of opponents of player i in general has multiple minimax strategies. The value of the skill measure depends on the selected minimax strategy. Although it does not influence the expected gains of player i as an optimal player, it does influence these numbers for this player as a beginner and as a fictive player. Borm and Van der Genugten (2001) solved this problem by replacing the minimax strategy with an approximation obtained by fictitious play with prescribed accuracy and starting with the strategy profile consisting of beginners' strategies. However, from a numerical point of view this is not a simple solution and can still be judged as a drawback of the concept.

Again, each game without external chance elements is a game of skill by definition. This is a consequence of the fact that the effects of the use of mixed strategies, the so-called internal chance elements, are not taken into account. The following example serves as an illustration.

Example 3.1 (Matrix game) Consider the following zero-sum game for two players. Both players have a coin. They simultaneously put their own coin on a table and cover it with one hand. The players can choose which side of the coin will be up, H(eads) or T(ails). If both players decide the same, then player 1 receives one dollar from his opponent. Otherwise, player 1 has to pay one dollar to player 2. The matrix A below summarizes the expected gains of player 1, the row player.

$$A = \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

This two-person zero-sum game has no external chance moves. Therefore, the random effect is equal to zero. According to the definition of Borm and Van der Genugten (2001), the consequence is a skill level of one, whatever the learning effect may be. However, in practice in this game players will always strongly randomize between the strategies available to them; this is pure gambling. Anyone observing this game will intuitively associate this randomization with a game of chance.

The message of this example is that merely the fact that optimal play needs randomization should influence the measure of relative skill. The following alternative definition of relative skill for more-person games incorporates this idea.

We drop the earlier idea of a fixed and uniform reference of the opponents against each type of player in a specific player role. Instead we let the opponents react optimally, depending on the type of player. Playing optimally must be interpreted as giving maximal opposition. This assumption on the behaviour of the coalition of opponents is only reasonable for zero-sum games. After all, for a zero-sum game, the coalition's aggregate gains are higher as the gains of player i are lower (and vice versa), while this relation does not hold for nonzero-sum games. This is not really a restriction, since any practical casino game you can think of can, maybe apart from some entrance fee, be modelled as a zero-sum game. If a bank (or dealer) is involved, this person should be considered as an extra player with only one (dummy) strategy.

We will give the formal definitions. Let $X_{-i} = \prod_{j \neq i} X_j$ denote the pure (coalition) strategies of the opponents of player i. Then for player i as *beginner* with strategy $\sigma_i^0 \in \Delta(X_i)$

the gains with optimal play of opponents are

$$U_i^0 := \min_{x_{-i}\in X_{-i}} U_i(\sigma_i^0, x_{-i}).$$

The expected gains for player i as an *optimal player* are given by his expected gains in the Nash equilibrium of the related two-person zero-sum game against the coalition of the other players:

$$U_i^m := \max_{\sigma_i\in\Delta(X_i)} \min_{x_{-i}\in X_{-i}} U_i(\sigma_i, x_{-i}) = \min_{\sigma_{-i}\in\Delta(X_{-i})} \max_{x_i\in X_i} U_i(x_i, \sigma_{-i}).$$

Note that the equality follows from the minimax theorem of Von Neumann (1928) and that U_i^m is exactly the value of the two-person zero-sum game. For player i as a *fictive player* we assume that he does not only know the outcome of the chance moves, but also the outcome of the randomization process of his opponents. This is the key change to the better incorporation of the randomization of the players in the definition of relative skill. So, in optimal play against a fictive player randomization has no effect at all. Therefore, the opponents will choose a pure strategy from X_{-i}, minimizing the maximum gains of a fictive player i. Player i will choose a strategy from $\bar{X}_i$ that maximizes his expected gains, given the strategy of his opponents. This leads to the expected gains U_i^f of the fictive player:

$$U_i^f := \min_{x_{-i}\in X_{-i}} \max_{\bar{x}_i\in\bar{X}_i} U_i(\bar{x}_i, x_{-i}).$$

It is not difficult to see that, for a specific player i, just as in the one-person case, we have for the ordering of expected gains of the different player types that $U_i^0 \leq U_i^m \leq U_i^f$. To find the expected gains of the game for each player type, we take the average over all n possible player roles. For the beginners, this leads to $U^0 = \frac{1}{n}\sum_{i=1}^n U_i^0$. Similarly, we have $U^m = \frac{1}{n}\sum_{i=1}^n U_i^m$ and $U^f = \frac{1}{n}\sum_{i=1}^n U_i^f$ for the expected gains for optimal and fictive players respectively.

The learning effect is again the difference between the expected gains for beginners and optimal players: $LE = U^m - U^0$. The contribution of player i to this learning effect is $\frac{1}{n}(U_i^m - U_i^0)$. Analogously, the random effect of the game is $RE = U^f - U^m$, with $\frac{1}{n}(U_i^f - U_i^m)$ as contribution of player i. Now we are ready to give the extension of the measure of skill for more-person games. Analogous to the measure for one-person games we define

$$RS = \frac{LE}{LE+RE} = \frac{U^m - U^0}{U^f - U^0} = \frac{\frac{1}{n}\sum_{i=1}^n (U_i^m - U_i^0)}{\frac{1}{n}\sum_{i=1}^n (U_i^f - U_i^0)}.$$

Let us illustrate the formulas with the matrix game from example 3.1.

Example 3.2 (Matrix game (2)) In this example we show how to calculate the skill measure for the matrix game we defined in example 3.1, starting from a certain characterization of beginner's play. Recall that the payoff matrix A of the game is as follows.

$$A = \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \left(\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \end{array}$$

Both players can choose from the same set of pure strategies: $X_1 = X_2 = \{H, T\}$. Beginners will probably choose between H and T randomly, so $\sigma_1^0 = \sigma_2^0 = \frac{1}{2}H + \frac{1}{2}T$. To compute the expected gains of the a beginner in the role of player 1, we check what his expected gains are if player 2 plays optimal against σ_1^0. Player 2 can choose either strategy to obtain a payoff of 0. Therefore, $U_1^0 = 0$ and, because of symmetry, $U_2^0 = 0$.

To compute the expected gains of the optimal players, U_1^m and U_2^m, we compute the Nash equilibrium of the matrix game. It is not difficult to see that this equilibrium is unique and that the for each player the equilibrium strategy is equal to the beginner's strategy. The value of the game, $v(A)$, is zero, so the optimal players have expected gains $U_1^m = U_2^m = v(A) = 0$.

In the fictive situation that player 1 can observe the outcome of the possible randomization of his opponent, he can always put his coin with the same side up. So, his gains will be 1, independent of the strategy choice of player 2. The same reasoning holds for player 2 as a fictive player, so we have $U_1^f = U_2^f = 1$.

Using these numbers, we can now compute the learning effect and the random effect for our matrix game.

$$\begin{array}{rcl} LE & = & \frac{1}{2}\sum_{i=1}^{2}(U_i^m - U_i^0) = \frac{1}{2}((0-0)+(0-0)) = 0 \\ RE & = & \frac{1}{2}\sum_{i=1}^{2}(U_i^f - U_i^m) = \frac{1}{2}((1-0)+(1-0)) = 1 \end{array}$$

The last step is to combine these effects to find the value of the skill measure:

$$RS = \frac{LE}{LE + RE} = \frac{0}{0+1} = 0.$$

Thus, following the new definition, we conclude that this is a pure game of chance. We stress, however, that changes in the assumptions on the behaviour of the beginners may influence the value of the skill measure. Note that the expected gains of the optimal players do not influence the value of RS in a two-person zero-sum game, since these values cancel out in the formula. However, if we want to see how much the players individually contribute to the learning effect and the random effect, we need to know the values of U_1^m and U_2^m. For this reason we are still interested in the value of the game.

After this small illustration of the computations the skill analysis requires, we will turn to the analysis of two more interesting games in the following section.

4 Examples

In this section we will illustrate the computations of the skill measure for more-person games that we presented in section 3 with two simple, but realistic games. Section 4.1 discusses a coin game. Subsequently, in section 4.2 we analyze the skill involved in a simple poker game among three players in which an external chance element, the dealing of cards, plays a role.

4.1 The coin game

In this section we consider a generalization of the n-coin game that was explored by Schwartz (1959). The n-coin game is a two-person zero-sum game in which both players have n coins available to play with. In the game we will discuss, this number of coins is not necessarily the same for both players. We call this generalization the (m, n)-coin game. Both players know the number of coins available to the opponent.

First, each player takes a number of coins, possibly zero, in his hand. A player cannot see what the opponent has taken in his hand. Now, first player 1 guesses the total number of coins taken by both players and then player 2 does the samem Subsequently, both players show their hands, so that this total can be determined. The game is won by the *first* player who guessed the total number of coins correctly. The winner receives a dollar from the opponent. If neither of the two guesses the right number, then nothing is paid.

We calculate the relative skill RS of the $(1, 2)$-coin game. A pure strategy of player 1 has the form $(i; j)$ where i is the number of coins he takes and j represents the sum he guesses. Player 2 plays a strategy of the form $(k; l_0, l_1, l_2, l_3)$. In this notation, k denotes the number of coins player 2 takes, while l_j tells us what number player 2 will guess if player 1 guessed the total number of coins to be j.

To find the value of this game, we have to find its normal form. Since the normal form is rather large, we restrict attention to the *quasi-reduced normal form.* Two pure strategies x_i and x_j of a player are called *realization equivalent* if they lead to the same terminal node for every given specification of the strategies for the other player, i.e. if x_i and x_j differ only at irrelevant information sets. The quasi-reduced normal form considers for player i a subset Q_i of his collection of pure strategies X_i, such that no two elements of Q_i are realization equivalent. If we construct this quasi-reduced normal form for the (1,2)-coin game and if we next iteratively delete dominated strategies from it, then we see that the game is equivalent to the reduced normal form game that is displayed in table 4.1. Player 1 is the row player, while the columns of the table correspond to pure strategies of player 2.

		Player 2											
		(0; 1, 0, 0, 0)	(0; 1, 0, 0, 1)	(0; 1, 0, 1, 0)	(0; 1, 0, 1, 1)	(1; 1, 2, 1, 1)	(1; 1, 2, 1, 2)	(1; 2, 2, 1, 1)	(1; 2, 2, 1, 2)	(2; 2, 2, 3, 2)	(2; 2, 3, 3, 2)	(2; 3, 2, 3, 2)	(2; 3, 3, 3, 2)
	(0; 0)	1	1	1	1	-1	-1	0	0	-1	-1	0	0
	(0; 1)	-1	-1	-1	1	1	1	1	1	-1	0	-1	0
	(0; 2)	-1	-1	0	0	-1	-1	-1	-1	1	1	1	1
Player 1	(0; 3)	-1	0	-1	0	-1	0	-1	0	-1	-1	-1	-1
	(1; 0)	-1	-1	-1	-1	0	0	-1	-1	0	0	-1	-1
	(1; 1)	1	1	1	1	-1	-1	-1	-1	0	-1	0	-1
	(1; 2)	0	0	-1	-1	1	1	1	1	-1	-1	-1	-1
	(1; 3)	0	-1	0	-1	0	-1	0	-1	1	1	1	1

Table 4.1: The reduced normal form of the (1, 2)-coin game.

It is easily verified that, by playing $\frac{1}{5}(0;0) + \frac{2}{5}(0;2) + \frac{2}{5}(1;2)$, player 1 can guarantee a value of $-\frac{1}{5}$. Analogously, player 2 can guarantee this value by playing $\frac{2}{5}(0;1,0,0,1) + \frac{1}{5}(1;1,2,1,2) + \frac{2}{5}(2;2,3,3,2)$. Therefore, the value of this game is $-\frac{1}{5}$. Then we know the expected gains for the optimal players, both in the role of player 1 and player 2: $U_1^m = -U_2^m = -\frac{1}{5}$.

The next step is to determine the strategies for the beginners. For player 1 two pure strategies can be considered "unreasonable" at first sight. After all, choosing (0; 3) or (1; 0) ensures him that he will not win the game, because the sum guessed is not reachable with his own choice of coins. Furthermore, randomization is a logical thing to do. Therefore, we assume that a naive player 1 chooses each of the remaining six pure strategies with equal probability. A possible best reply of player 2 is the pure strategy (1; 1, 2, 1, 2). This results in the expected gains of player 1 as a beginner: $U_1^0 = -\frac{1}{3}$.

For player 2 there is a number of strategies that is immediately seen to be irrelevant. It makes no sense for player 2 to guess a sum that is smaller than the number of coins he has in hand. Even someone who plays the game for the first time will see this. Consequently, all strategies with $l_j < k$ for any j are left out of consideration. A beginner in the role of player 2 will play a fair randomization over the remaining strategies. A best response of

player 1 against this strategy σ_2^0 is to play (0;0). In this way player 2 has to pay an expected value of $\frac{71}{109}$. So $U_2^0 = -\frac{71}{109}$. Taking a fair average over the player roles yields us expected beginner's gains of $U^0 = -\frac{161}{327}$.
Computation of the expected payoffs of fictive players is easy. A fictive player knows how many coins the opponent has taken in his hand. With this information, player 1 can always win the game by guessing the sum of this number and the (arbitrarily chosen) number of coins in his own hand. There is nothing player 2 can do to prevent him from winning. Player 2 can do something similar, but he needs to have at least one coin to make sure he can form a sum of coins that player 1 has not guessed yet. Since player 2 has two coins available, as a fictive player he will succeed in winning too. Hence, the expected gains for the fictive players are $U_1^f = U_2^f = 1$. Consequently, the average expected gains for a fictive player are $U^f = 1$.

We can now compute the learning effect and the random effect for the $(1, 2)$-coin game.

$$\begin{aligned} LE &= \tfrac{1}{2}\textstyle\sum_{i=1}^{2}(U_i^m - U_i^0) = \tfrac{1}{2}((-\tfrac{1}{5}+\tfrac{1}{3})+(\tfrac{1}{5}+\tfrac{71}{109})) = \tfrac{161}{327} \\ RE &= \tfrac{1}{2}\textstyle\sum_{i=1}^{2}(U_i^f - U_i^m) = \tfrac{1}{2}((1+\tfrac{1}{5})+(1-\tfrac{1}{5})) = 1 \end{aligned}$$

The last step is to combine these effects to find the value of the skill measure:

$$RS = \frac{LE}{LE + RE} = \frac{161}{488} \approx 0.33.$$

The resulting skill value of 0.33 strongly depends on our definition of how beginners act. If the explanation of the (1,2)-coin game to a new player includes an advise on strategy selections, then a starting player may do significantly better than "our" beginner and, consequently, the skill level we find for this game would be lower.

The next section gives an illustration of the way the skill measure should be computed for games with more than two players in which an external chance element is involved.

4.2 Drawpoker

In this section we consider the simplified version of standard drawpoker that is discussed by Binmore (1992). The first simplification is that there is no second dealing round in which players can change a number of cards. The second simplification is that we do not use a standard deck with the usual types of hands of five cards. Instead of that, all cards have distinct values and every player gets only one card, dealt with or without replacement. In this way we get a poker game which still contains most of the strategic features of standard drawpoker. First we will give a formal description of the game and thereafter we will compute the relative skill of the three-person version of this game that Binmore (1992) also analyzed. For general references on specific aspects of poker, we refer to Epstein (1977) and Scarne (1990).

We will describe the rules of the most general form of our version of drawpoker precisely. The game is played with n players, numbered from 1 to n, where $n \geq 2$. At the beginning of the game, each player places the initial bet (or ante) a into the stakes. Then each player gets one card. This card is randomly chosen out of a deck of c cards with card values $0, 1, \ldots, c-1$. According to the set-up of the game, cards are dealt with ($r = 1$) or without ($r = 0$) replacement.

Then the betting starts, beginning with player 1, followed by player 2 and so on. Player 1 can *pass* or open the game with *bet*. If he bets, then he chooses an amount b_i from a given set of s betting possibilities $b_1, \ldots, b_s$ and places this into the pot. As long as the game is not opened with a bet, the next players have the same choice of moves. If all players have passed, then the game is a draw and the ante is returned to the players.

As soon as the game is open, the player whose turn it is can choose between three actions. He may *call*, *raise* or *fold*. His successors have the same choice. A call means placing an amount into the pot equal to that of the last bet made. A raise means that the last bet is raised with some extra amount chosen from the set of betting possibilities and this new bet is placed into the pot. A fold means that the player drops out of the game and loses his contribution to the stakes.

The total number of raises, including the first bet, must not exceed a certain maximum m. A player cannot raise or call again on his own bet or raise. So the betting round ends if at a stage where the player who's turn it would be next is the last player who did not call or fold. If at such a stage, this player is the only player still in the game, then he wins the pot. Otherwise, a showdown is forced. In a showdown all players that are still in the game show their cards. The player with the highest card wins the pot. If more than one player has the highest card, which is only possible if cards were dealt with replacement, then the pot is equally divided between them.

To compute the skill level of our drawpoker game, we have to specify the strategy of a beginner in all possible player roles. It is not easy to judge how a beginner would play this game in all variants. Perhaps a very simple way is to imagine that he has two card values in mind: a raisecard RC and a foldcard FC. He uses these values to play as follows.

- If his card value is less than or equal to FC, then he folds whenever this choice is available to him, otherwise he passes.

- If his card is greater than or equal to RC, then he raises (or bets, after a pass) the maximum, b_s.
- If his card is between FC and RC, then he does not pass, but bets the minimum, b_1. If the game is already open, then he calls.

Of course the values of FC and RC will depend on the game parameters. It seems reasonable to assume that a beginner bets or raises if, roughly spoken, his card value is among the highest 10% in the deck. With respect to the determination of the foldcard, we think that the beginner will in general play with too much opportunism and will not pass or fold unless his card is among the lowest 50% in the deck. If the number of players becomes larger, this percentage may increase, but it will probably not exceed 70%. This choice is deflected in the following formula for FC. Clearly, this is too difficult to compute for a beginner, but the resulting value for the foldcard satisfies the preceding description of the behaviour of our beginners.

$$\begin{aligned} FC &= \max\left(0, (c-2) - \left\lfloor (c-1)\left(\tfrac{1}{2} - \tfrac{1}{10}\ln(n-1)\right)\right\rfloor\right) \\ RC &= (c-1) - \left\lfloor \tfrac{1}{10}(c-1)\right\rfloor \end{aligned}$$

Here, $\lfloor x \rfloor$ denotes the integer part of x. This notion is used to make sure that the boundaries are given by (integer) card numbers.

So far we kept our notation and definitions with respect to the game as general as possible. In the remainder of this section we will restrict our attention to the variant for which Binmore (1992) computed the Nash equilibrium. This is the 3-person drawpoker game with 2 cards, L (low) and H (high), that are dealt *with* replacement ($r = 1$). The ante is $a = 2$ and the only possible betting amount is $b_0 = 8$. Only one bet is allowed; when the game is open, players are only allowed to fold or call. This parameter choice enables us to do a large part of the skill analysis manually. Binmore already showed that the pure strategy space for all three players can be reduced enormously.

For this game the beginners' strategies we proposed above boil down to passing or folding with an L and raising or calling with an H.

For our analysis we are interested in the expected gains of the three players when the coalition formed by the two opponents gives maximal opposition. Let us consider the two-person zero-sum game that corresponds to the situation in which player 1 plays against the coalition of the players 2 and 3. After elimination of dominated strategies player 1 has two pure strategies left: $X_1 = \{x_1, x_2\}$. Strategy x_2 is the beginner's strategy, i.e. passing or folding with an L and raising or calling with an H. The other pure strategy, x_1, differs from x_2 in one position; it prescribes passing with a high card. This phenomenon in which a player with a good hand tries to mislead his opponents is called *sandbagging*. The coalition of players 2 ando3 also have only two undominated pure strategies: $X_{23} = \{y_1, y_2\}$. Translation of these strategies in terms of strategies for the two individual players in this coalition shows that player 2 always plays the same strategy as the beginner. Player 3 does that too in strategy y_1, but in y_2 he bets with a low card when neither of his predecessors has opened the game. This strategic aspect of poker is more familiar than sandbagging; this is called *bluffing*. The matrix $A_{1,23}$ below displays the net expected gains for player 1 in this reduced two-person zero-sum game.

$$A_{1,23} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} -\frac{1}{2} & \frac{3}{4} \\ 0 & -\frac{1}{4} \end{pmatrix} \end{array}$$

In the unique Nash equilibrium of this game player 1 plays $\frac{1}{6}x_1 + \frac{5}{6}x_2$, while his opponents use the mixed strategy $\frac{2}{3}y_1 + \frac{1}{3}y_2$. The resulting game value is $v(A_{1,23}) = -\frac{1}{12}$. Therefore, the expected gains for optimal player 1 is $U_1^m = -\frac{1}{12}$.

When player 1 plays the strategy of a beginner, then a possible best answer of the coalition is to let player 2 play the beginner's strategy and let player 3 play the bluffing strategy, i.e. to bet with an L when the other players have not opened the game. Since player 1 already reveals his card in the opening move, the other players make sure he does not gain anything unnecessary with an L and that the coalition does not lose any betting amounts if player 1 has an H. In this way player 1 makes an expected loss of $\frac{1}{4}$ and therefore his expected gains are $U_1^0 = -\frac{1}{4}$.

Now, let us see what the possibilities of the fictive player are. He knows the cards of the other players and the opponents know this. Therefore, for players 2 and 3 bluffing and sandbagging become useless if they face a fictive player. So, if player 1 knows that neither player 2 nor player 3 has a higher card than he has, the best he can do with an L is to bet (bluff). The opponents will fold. If he has a high card, he will always bet. This is how player 1 as a fictive player can make expected gains of $U_1^f = \frac{1}{2}$.

For the other players similar computations have to be carried out. Then we find the values that are displayed in table 4.2. From the numbers in this table we can compute the learning effect and the random effect for 3-person drawpoker.

$$\begin{aligned} LE &= \tfrac{1}{3}\textstyle\sum_{i=1}^{3}(U_i^m - U_i^0) = \tfrac{1}{3}((-\tfrac{1}{12} + \tfrac{1}{4}) + (-\tfrac{1}{12} + \tfrac{1}{4})) + (\tfrac{1}{10} - 0)) = \tfrac{13}{90} \\ RE &= \tfrac{1}{3}\textstyle\sum_{i=1}^{3}(U_i^f - U_i^m) = \tfrac{1}{3}((\tfrac{1}{2} + \tfrac{1}{12}) + (\tfrac{1}{2} + \tfrac{1}{12}) + (\tfrac{1}{2} - \tfrac{1}{10})) = \tfrac{47}{90} \end{aligned}$$

	player 1	player 2	player 3
beginner	$-\frac{1}{4}$	$-\frac{1}{4}$	0
optimal	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{10}$
fictive	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 4.2: The expected gains in 3-person drawpoker.

The last step is to combine these effects to find the value of the skill measure.

$$RS = \frac{LE}{LE + RE} = \frac{13}{60} \approx 0.22$$

Just as in the coin game example, we would like to stress that this result depends on our definition of beginners' behaviour. If this game is played in a casino where the brochure with the game description ends with some words on bluffing and sandbagging, then a beginner may use a strategy that differs heavily from the one we assumed him to use. This may also affect the skill level we find for the drawpoker game.

5 Concluding remarks

In this paper we presented a modification of the skill measure of Borm and Van der Genugten (2001) for more-person games. The new measure takes into account the effect of internal chance moves, the chance effect that is caused by the players themselves by randomization in choosing their actions. The result is a unique number indicating the skill level, that can be computed for a large class of games. In this final section we want to make two remarks with respect to the computations. The first is about the behaviour of naive players and the second is about measuring the game result of a player.

One of the numbers used in the definition of skill, is the expected gain of a beginner. To compute this number, we have to make assumptions about the strategies selected by naive players. Although these strategies strongly depend on game specific aspects, it is possible to give a few rules of thumb that can be used for determination of the behaviour of the beginning players. Practical situations indicate that, in general, beginners do not succeed in combining safety and risk into an appropriate strategy. Furthermore, beginners have difficulties taking into account the situation of the opponents. They focus heavily on their own situation. Think of a poker game; beginners mostly ignore cards dealt to other players and base their decisions only on their own hand. Another strategic aspect that is difficult to apply for a beginner is randomization. As a consequence, players who have just mastered the rules of a game tend to use pure strategies.

The qualification of beginners' behaviour in specific games changes over time. When the popularity of a game increases, the "news" about strategic aspects will spread quickly and players can already gain from experience—although not their own—the first time they participate in the game. A related issue is the way a game is presented. Recall the example of the poker brochure that gives information on bluffing and sandbagging to see that even the presentation of the game may affect the behaviour of beginning players, and thus influences the game's skill value.

Example 3.2 demonstrated that the skill measure is applicable to the class of matrix games too. However, one should be careful with conclusions based on the resulting value of the skill measure. After all, the definition of a beginner's strategy is a very subjective element here and may very well depend on "the story behind the matrix".

The second aspect on which we want to make a remark is the measurement of the quality of a specific strategy of a player. We used the expected gains corresponding to the strategies to compare them, but this is not the only possibility. Another option would be the mean gains per bet in the long run. Alternative ways of expressing strategy quality in our definitions will have our attention in future research. At first sight, however, we think that the notion of expected gains is an appropriate choice, because this is the what players seem to (try to) maximize in practice.

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